Lecture 2 Intro to Heat Flow

Surface heat flow

Heat flux from the Sun (mostly reradiated): 400 W/m^2
Heat flux from Earth's interior: 80 mW/m^2
Earthquake energy loss: 0.2 mW/m^2
Heat flow from human?

- energy intake: 2000 "calories" ≈ 8000 kJ (W = J/s)
- 8000 kJ / 24 hr = ~100 J/s = 100 W (1 day ≈ 80,000 s)
- surface area: 2 m x 1 m = 2 m^2
- 50 W/m^2 ! — or one lightbulb

Types of Heat Transport

conduction
convection
radiation—electromagnetic radiation
advection

Relationship Between Heat Flow & T Gradient: Fourier's Law

The rate of heat flow is proportional to the difference in heat between two bodies. A thin plate of thickness z with temperature difference ΔT experiences heat flow Q:

$$Q = -k\frac{\Delta T}{z} \qquad \textit{units: } W/m^2 \textit{ or } J/m^2 s$$

where k is a proportionality constant called the *thermal conductivity* (J/msK):

Ag	418
rock	1.7–3.3
glass	1.2
wood	0.1

We can express the above equation as a differential by assuming that $z \to 0$:

$$Q(z) = -k\frac{\partial T}{\partial z} \qquad \textit{units}: \frac{J}{m^2 s} = -\frac{J}{msK}\frac{K}{m}$$

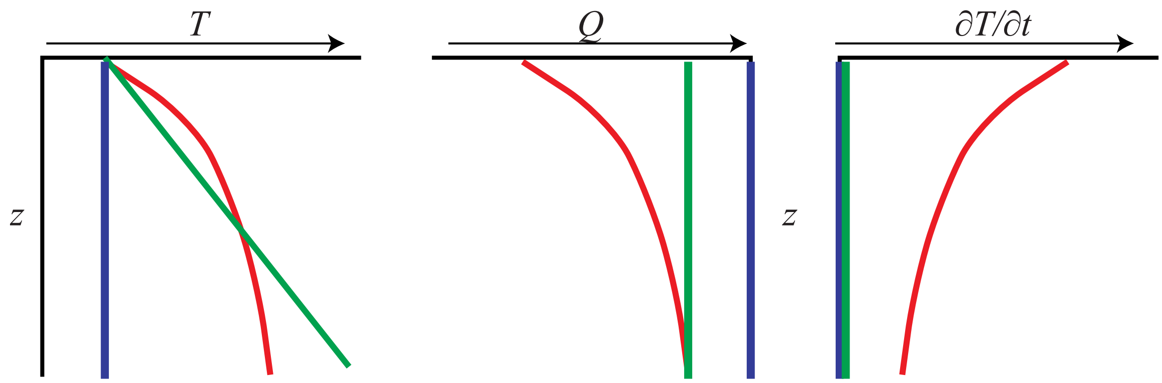
(We use a minus sign because heat flows from hot to cold and yet we want positive T to correspond to positive x, y, z.)
In other words, the heat flow at a point is proportional to the local slope of the T–z curve (the geotherm).
If the temperature is constant with depth ($\partial T/\partial z = 0$), there is no heat flow—of course!
Moreover, if $\partial T/\partial z$ is constant (and nonzero) with depth ($T(z)=T_{zo}+mz$), the heat flow will be constant with depth; this is clearly a steady state.

Generalized to 3D, the relationship between heat flow and temperature is:

$$Q = -k\nabla T = -k\left(\frac{\partial T}{\partial x} + \frac{\partial T}{\partial y} + \frac{\partial T}{\partial z}\right)$$

i.e., the heat flow at a point is proportional to the local temperature gradient in 3D.

Relationship Between T Change and T Gradient: The Diffusion Equation



Of course, if the heat flow is *not* constant with depth, the temperature *must* be changing. The temperature at any point changes at a rate proportional to the local gradient in the heat flow:

$$\frac{\partial T}{\partial t} = -\frac{1}{\rho C_P}\frac{\partial Q}{\partial z} \qquad units: \ \frac{K}{s} = \frac{1}{\left(kgm^{-3}\right)\left(J/kgK\right)}\frac{J/m^2 s}{m}$$

So, if there is no *gradient* in the heat flow ($\partial Q/\partial z = 0$), the temperature does not change. If we then stuff the equation defining heat flow as proportional to the temperature gradient ($Q = -k\ \partial T/\partial z$) into the equation expressing the rate of temperature change as a function of the heat flow gradient ($\partial T/\partial z\ \alpha\ \partial Q/\partial z$), we get the rate of temperature change as a function of the curvature of the temperature gradient (perhaps more intuitive than the previous equation):

$$\frac{\partial T}{\partial t} = \frac{k}{\rho C_P}\frac{\partial^2 T}{\partial z^2} \qquad units: \ \frac{K}{s} = \frac{J/msK}{\left(kgm^{-3}\right)\left(J/kgK\right)}\frac{K}{m^2}$$

And, in 3D, using differential operator notation (∇^2 is known as 'the Laplacian'):

$$\frac{\partial T}{\partial t} = \frac{k}{\rho C_P}\nabla^2 T$$

This is the famous 'diffusion equation'. Wheee! It can be expressed most efficiently as

$$\frac{\partial T}{\partial t} = \kappa \nabla^2 T$$

where κ is the thermal diffusivity (m/s^2):

$$\kappa = \frac{k}{\rho C_P}$$

Heat Production: The Heat Production Equation

Rocks are radiogenic (to varying degrees), so we need some way of incorporating heat generation. We will use *A* for heat generation per unit volume per unit time (W/m^3 or J/m^3s). This adds a term to the diffusion equation, giving the 'heat conduction equation':

$$\frac{\partial T}{\partial t} = \frac{k}{\rho C_P} \nabla^2 T + \frac{A}{\rho C_P}$$

Most of the heat generation in Earth is from the decay of ^{238}U, ^{235}U, ^{232}Th, and ^{40}K.

Radiogenic heat production ($\mu W/m^3$) of some rocks (from Fowler, *The Solid Earth*):

granite	2.5
average continental crust	1
tholeiitic basalt	0.08
average oceanic crust	0.5
peridotite	0.006
average undepleted mantle	0.02

Calculating a Simple Geotherm Given a Surface Heat Flux & Surface T

With no erosion or deposition and a constant heat flux, a steady-state thermal gradient can be established. By definition, at steady state

$$\frac{\partial T}{\partial t} = 0$$

and the heat conduction equation can then be simplified and re-arranged:

$$\frac{k}{\rho C_P} \frac{\partial^2 T}{\partial z^2} = -\frac{A}{\rho C_P} \quad \text{or} \quad \frac{k}{\rho C_P} \nabla^2 T = -\frac{A}{\rho C_P}$$

or:

$$\frac{\partial^2 T}{\partial z^2} = -\frac{A}{k} \quad \text{or} \quad \nabla^2 T = -\frac{A}{k}$$

in other words, the curvature of the geotherm is dictated by the heat production rate *A* divided by the thermal conductivity *k*. Pretty simple. To calculate the geotherm, we integrate the above equation, getting:

$$\frac{\partial T}{\partial z} = -\frac{A}{k} z + C_1$$

We can evaluate C_1 if we specify the surface heat flow, $Q_S = k\partial T/\partial z$, as a boundary condition at $z = 0$:

$$Q_S = kC_1 \quad \text{or} \quad C_1 = \frac{Q_S}{k}$$

Stuffing this back into the previous equation:

$$\frac{\partial T}{\partial z} = -\frac{A}{k} z + \frac{Q_S}{k}$$

and integrating a second time gives:

$$T = -\frac{A}{2k}z^2 + \frac{Q_S}{k}z + C_2$$

If the temperature at Earth's surface is T_S, $C_2 = T_S$. The geotherm is thus given by

$$T = -\frac{A}{2k}z^2 + \frac{Q_S}{k}z + T_S$$

where A is the volumetric heat production rate and Q_S is the surface heat flow.

Calculating a Simple Geotherm Given a Basal Heat Flux & Surface T

Let's calculate a geotherm dictated by a surface temperature and a basal (e.g., Moho) heat flux at depth z_M. We integrate once as above:

$$\frac{\partial T}{\partial z} = -\frac{A}{k}z + C_1$$

and, if we set $Q_M = k\partial T/\partial z$ at z_M as a boundary condition, then

$$\frac{Q_M}{k} = -\frac{A}{k}z + C_1 \qquad \text{or} \qquad C_1 = \frac{Q_M}{k} + \frac{A}{k}z$$

Stuffing this back into the previous equation:

$$\frac{\partial T}{\partial z} = -\frac{A}{k}z + \left(\frac{Q_M + Az_M}{k}\right)$$

and integrating a second time gives:

$$T = -\frac{A}{2k}z^2 + \left(\frac{Q_M + Az_M}{k}\right)z + C_2$$

If the temperature at Earth's surface is T_S, $C_2 = T_S$. The geotherm is thus given by

$$T = -\frac{A}{2k}z^2 + \left(\frac{Q_M + Az_M}{k}\right)z + T_S$$

where A is the volumetric heat production rate and Q_M is the basal heat flow at depth z_M. Note that this equation reveals that the basal heat flow contributes $Q_M z/k$ to the temperature at depth z.